5.9 Higher Order Equations and Systems of ODE.

Consider the System:

$$\begin{cases} u_1'(t) = -\frac{1}{2} u_1(t) + u_2(t) & (1.1) \\ u_2'(t) = -u_1(t) - \frac{1}{2} u_2(t) & (1.2) \end{cases}$$

ICs: $u_1(0) = 1, \quad u_2(0) = 1, \quad t \in [0, b]$. (1.3)

Extension of R-K 4th order to System (1.1)-(1.3).

More general,

$$\begin{cases} u_1'(t) = f_1(t, u_1, u_2), \\ u_2'(t) = f_2(t, u_1, u_2), \end{cases} \qquad t \in [0, b]$$

$$u_1(0) = \alpha, \quad u_2(0) = \beta.$$

Discussion on Conditions for existence and Uniqueness later.

Eigenvalues

$r_1 = -\frac{1}{2}+i, \quad r_2 = -\frac{1}{2}-i$

Eigenvectors

$\vec{\xi}^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

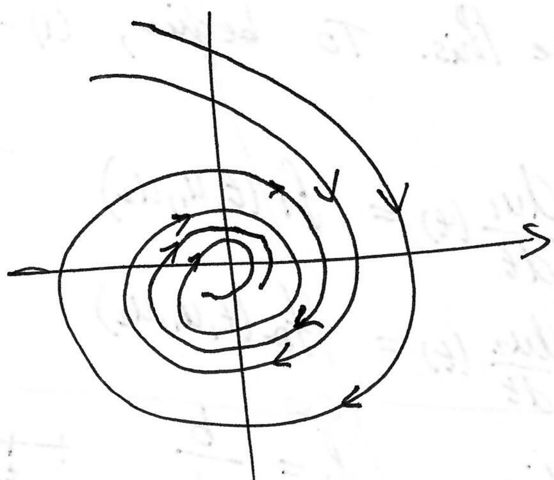
$\vec{u}(t) = \vec{\xi} e^{rt}$ $S_{1,1}$

$$\vec{u}(t) = C_1 e^{-t/2} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{u}(0) = C_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{matrix} C_1 = 1 \\ C_2 = 1 \end{matrix}$$

$$\vec{u}(t) = e^{-t} \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + e^{-t/2} \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

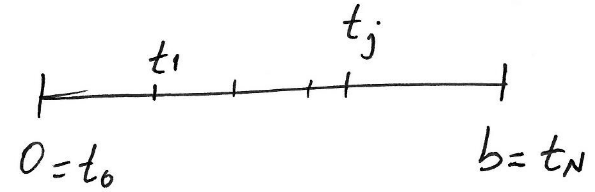
Phase portrait:



$$\vec{u}'(t) = A \vec{u}(t)$$

$$r \vec{\xi} \cancel{e^{rt}} = (A \vec{\xi}) \cancel{e^{rt}} \Rightarrow A \vec{\xi} = r \vec{\xi}$$

$$\begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = r \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$



$$h = \frac{b-a}{N}$$

$t_j = a + jh$, $j = 0, \dots N.$ $\quad w_{i,j} = u_i(t_j)$, $\quad i = 1,2$, $j = 1,2,.., N$

$w_{1,0} = \alpha$, $\quad w_{2,0} = \beta$

R-K 4th order:

$$\begin{cases} w_{1,j+1} = w_{1,j} + \frac{1}{6}\left[K_{1,1} + 2K_{2,1} + 2K_{3,1} + K_{4,1}\right] \\ w_{2,j+1} = w_{2,j} + \frac{1}{6}\left[K_{1,2} + 2K_{2,2} + 2K_{3,2} + K_{4,2}\right] \\ w_{1,0} = \alpha, \quad w_{2,0} = \beta \qquad\qquad j = 0,1,2,.., N-1. \end{cases}$$

Where $K_{1,i} = h f_i(t_j, w_{1,j}, w_{2,j})$, $\quad i = 1,2$

$$K_{2,i} = h f_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}K_{1,1}, w_{2,j} + \frac{1}{2}K_{1,2}\right)$$

$$K_{3,i} = h f_i\left(t_j + \frac{h}{2}, w_{1,j} + \frac{1}{2}K_{2,1}, w_{2,j} + \frac{1}{2}K_{2,2}\right)$$

$$K_{4,i} = h f_i\left(t_j + h, w_{1,j} + K_{3,1} + w_{2,j} + K_{3,2}\right)$$

$i = 1,2$

In our example,

$$\begin{cases} f_1(t, u_1, u_2) = -\frac{1}{2}u_1 + u_2 \\ f_2(t, u_1, u_2) = -u_1 - \frac{1}{2}u_2 \end{cases}$$

$$f_i(t_j, w_{1,j}, w_{2,j}) \simeq f_i(t_j, u_1(t_j), u_2(t_j)), \quad i=1,2$$

Then,

$$k_{1,1} = h\, f_1(t_j, w_{1,j}, w_{2,j}) = h\left[-\frac{1}{2}w_{1,j} + w_{2,j}\right]$$

$$k_{1,2} = h\, f_2(t_j, w_{1,j}, w_{2,j}) = h\left[-w_{1,j} - \frac{1}{2}w_{2,j}\right]$$

$$K_{2,1} = h\, f_1\left(t_j + h/2,\ w_{1,j} + \frac{1}{2}k_{1,1},\ w_{2,j} + \frac{1}{2}k_{1,2}\right) = h\left[-\frac{1}{2}\left(w_{1,j} + \frac{1}{2}k_{1,1}\right) + w_{2j} + \frac{1}{2}k_{1,2}\right]$$

$$K_{2,2} = h\, f_2\left(t_j + h/2,\ w_{1,j} + \frac{1}{2}k_{1,1},\ w_{2,j} + \frac{1}{2}k_{1,2}\right) =$$

$$= h\left[-\left(w_{1,j} + \frac{1}{2}k_{1,1}\right) - \frac{1}{2}\left(w_{2,j} + \frac{1}{2}k_{1,2}\right)\right]$$

etc. . . .

For more general definition review pp. 315-317. ~~book~~ B-F.

Existence and Uniqueness thm.

A mth-order system has the form:

$$\frac{du_1}{dt}(t) = f_1(t, u_1, u_2, .., u_m) \qquad (3)$$

$$\vdots$$

$$\frac{du_m}{dt}(t) = f_m(t, u_1, u_2, .., u_m), \qquad a \le t \le b$$

IC's are $u_1(a) = \alpha_1, \ .., \ u_m(a) = \alpha_m \qquad (4)$

Def - (Lipschitz cond.)

$f(t, y_1, .., y_m)$ defined on $D = \left\{ (t, u_1, .., u_m) \mid a \le t \le b, \ -\infty \le u_i \le \infty, \ i = 1, 2, .., m \right\}$

is said to satisfy a Lipschitz cond. on D in the variables $u_1, ..., u_m$, if there is $L > 0$ s.t.

$$|f(t, u_1, .., u_m) - f(t, z_1, .., z_m)| \le L \sum_{j=1}^{m} |u_j - z_j| = \|\vec{u} - \vec{z}\|_1$$

It might be any other norm.

for all pair $(t, u_1, .., u_m)$ and $(t, z_1, .., z_m) \in D$.

$$f: D \subset \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$$

Proposition:
- f and $\frac{\partial f}{\partial u_i}$ are conts on D, $i=1,2,..,m$
- $$\left|\frac{\partial f}{\partial u_i}(t, u_1, .., u_m)\right| \le L \qquad i=1,2,..,m$$

for all $(t, u_1, .., u_m) \in D$

then f satisfies Lipshitz cond on D with Lipschitz constant L.

Thm 5.17
- $f_i(t, u_1, .., u_m)$ are conts on D. $i=1,2,..,m$
- f_i $i=1,2,..,m$ satisfy a Lipschitz cond. on D in $u_1, u_2, .., u_m$

then, the IVP (3)-(4) has a unique soln. $u_1(t), .., u_m(t)$, for $a \le t \le b$.

For our example in 2-D.

$$f_1(t, u_1, u_2) = -\frac{1}{2}u_1 + u_2, \qquad D = \{(t, u_1, u_2) \mid 0 \le t \le b, -\infty < u_i < \infty\}$$

$$f_2(t, u_1, u_2) = -u_1 - \frac{1}{2}u_2,$$

$$\frac{\partial f_1}{\partial u_1} = -\frac{1}{2}, \quad \frac{\partial f_1}{\partial u_2} = 1, \quad \frac{\partial f_2}{\partial u_1} = -1, \quad \frac{\partial f_2}{\partial u_2} = -\frac{1}{2}$$

$$\left|\frac{\partial f_1}{\partial u_i}\right| \le 1 \equiv L_1, \quad i=1,2$$

$$\left|\frac{\partial f_2}{\partial u_i}\right| \le 1 \equiv L_2, \quad i=1,2.$$

f_1, f_2 Satisfy Lipschitz cond in u_1, u_2 and are conts on D. $\Rightarrow$ There is a unique soln on $[0,1]$.

5.9?

Higher order IVP $\begin{cases} y^{(m)}(t) = f(t, y, y', \ldots, y^{(m-1)}) \\ y(a) = \alpha_1, y'(a) = \alpha_2, \ldots, y^{(m-1)}(a) = \alpha_m \end{cases}$, $a \le t \le b$

It can be written as an ODE syst. or sys

$\begin{cases} u_1 = y \\ u_2 = y' \\ \vdots \\ u_m = y^{(m-1)} \end{cases} \Rightarrow \begin{array}{l} u_1' = u_2 \\ u_2' = u_3 \\ \vdots \\ u_m' = y^{(m)} = f(t, y, y', \ldots, y^{(m-1)}) \\ = f(t, u_1, u_2, \ldots, u_m) \end{array}$

IC's:

$u_1(a) = \alpha_1$

$u_2(a) = \alpha_2$

$\vdots$

$u_m(a) = \alpha_m$

Example 2 Consider the second-order initial-value problem

$$y'' - 2y' + 2y = e^{2t} \sin t, \qquad \text{for } 0 \le t \le 1, \qquad \text{with } y(0) = -0.4,\ y'(0) = -0.6.$$

Let $u_1(t) = y(t)$ and $u_2(t) = y'(t)$. This transforms the equation into the system

$$\boxed{u_1'(t) = u_2(t),} \quad \boxed{u_2'(t) = e^{2t} \sin t - 2u_1(t) + 2u_2(t),}$$

$$\quad = f_1(t, u_1, u_2) \qquad\qquad = f_2(t, u_1, u_2)$$

with initial conditions

$$u_1(0) = -0.4, \quad u_2(0) = -0.6.$$

The Runge–Kutta fourth-order method will be used to approximate the solution to this problem using $h = 0.1$. The initial conditions give $w_{1,0} = -0.4$ and $w_{2,0} = -0.6$. Eqs. (5.48) through (5.51) with $j = 0$ give

$$k_{1,1} = hf_1(t_0, w_{1,0}, w_{2,0}) = hw_{2,0} = -0.06,$$

$$k_{1,2} = hf_2(t_0, w_{1,0}, w_{2,0}) = h[e^{2t_0} \sin t_0 - 2w_{1,0} + 2w_{2,0}] = -0.04,$$

$$k_{2,1} = hf_1\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right) = h[w_{2,0} + \frac{1}{2}k_{1,2}] = -0.062,$$

$$k_{2,2} = hf_2\left(t_0 + \frac{h}{2}, w_{1,0} + \frac{1}{2}k_{1,1}, w_{2,0} + \frac{1}{2}k_{1,2}\right)$$

$$= h\left[e^{2(t_0+0.05)} \sin(t_0 + 0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{1,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{1,2}\right)\right]$$

$$= -0.03247644757,$$

$$k_{3,1} = h\left[w_{2,0} + \frac{1}{2}k_{2,2}\right] = -0.06162832238,$$

$$k_{3,2} = h\left[e^{2(t_0+0.05)} \sin(t_0 + 0.05) - 2\left(w_{1,0} + \frac{1}{2}k_{2,1}\right) + 2\left(w_{2,0} + \frac{1}{2}k_{2,2}\right)\right]$$

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$$= -0.03152409237,$$

$$k_{4,1} = h\left[w_{2,0} + k_{3,2}\right] = -0.06315240924,$$

and

$$k_{4,2} = h\left[e^{2(t_0+0.1)}\sin(t_0 + 0.1) - 2(w_{1,0} + k_{3,1}) + 2(w_{2,0} + k_{3,2})\right] = -0.02178637298.$$

So

$$w_{1,1} = w_{1,0} + \tfrac{1}{6}(k_{1,1} + 2k_{2,1} + 2k_{3,1} + k_{4,1}) = -0.4617333423$$

and

$$w_{2,1} = w_{2,0} + \tfrac{1}{6}(k_{1,2} + 2k_{2,2} + 2k_{3,2} + k_{4,2}) = -0.6316312421.$$

The value $w_{1,1}$ approximates $u_1(0.1) = y(0.1) = 0.2e^{2(0.1)}(\sin 0.1 - 2\cos 0.1)$, and $w_{2,1}$ approximates $u_2(0.1) = y'(0.1) = 0.2e^{2(0.1)}(4\sin 0.1 - 3\cos 0.1)$.

The set of values $w_{1,j}$ and $w_{2,j}$, for $j = 0, 1, \ldots, 10$, obtained using the Runge–Kutta method of order four, are presented in Table 5.18 and are compared to the actual values of $u_1(t) = 0.2e^{2t}(\sin t - 2\cos t)$ and $u_2(t) = u_1'(t) = 0.2e^{2t}(4\sin t - 3\cos t)$. ■

5.18

$y(t_j) = u_1(t_j)$	$w_{1,j}$	$y'(t_j) = u_2(t_j)$	$w_{2,j}$	$\lvert y(t_j) - w_{1,j}\rvert$	$\lvert y'(t_j) - w_{2,j}\rvert$
−0.40000000	−0.40000000	−0.6000000	−0.60000000	0	0
−0.46173297	−0.46173334	−0.6316304	−0.63163124	3.7×10^{-7}	7.75×10^{-7}
−0.52555905	−0.52555988	−0.6401478	−0.64014895	8.3×10^{-7}	1.01×10^{-6}
−0.58860005	−0.58860144	−0.6136630	−0.61366381	1.39×10^{-6}	8.34×10^{-7}
−0.64661028	−0.64661231	−0.5365821	−0.53658203	2.03×10^{-6}	1.79×10^{-7}
−0.69356395	−0.69356666	−0.3887395	−0.38873810	2.71×10^{-6}	5.96×10^{-7}
−0.72114849	−0.72115190	−0.1443834	−0.14438087	3.41×10^{-6}	7.75×10^{-7}
−0.71814890	−0.71815295	0.2289917	0.22899702	4.05×10^{-6}	2.03×10^{-6}
−0.66970677	−0.66971133	0.7719815	0.77199180	4.56×10^{-6}	5.30×10^{-6}
−0.55643814	−0.55644290	1.534764	1.5347815	4.76×10^{-6}	9.54×10^{-6}
−0.35339436	−0.35339886	2.578741	2.5787663	4.50×10^{-6}	1.34×10^{-5}